ANALYSIS OF A THREE – phase induction motor Directly from Maxwell's equations

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Abstract

The torque developed in a 3Phase AC squirrel cage motor is generally expressed in terms of resistances and reactances of the stator, the rotor, and the motor as a whole. Here we use Maxwell's equations directly to find the torque in terms of geometrical parameters.

1. Introduction

The standard procedure[1] for evaluating the performance of a 3Phase AC squirrel cage motor is a perphase analysis of a circuit containing Thevenin equivalents of the stator, the rotor and the load. Analysis proceeds as in a transformer, with the coupling between windings taken as a function of the rotation speed. The circuit diagram is shown in Figure 1.

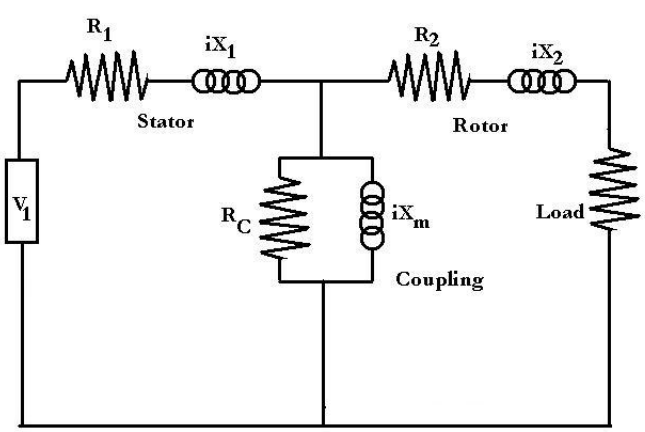

Figure 1 : Circuit Diagram of AC Motor

Here R_1 and X_1 refer to the resistance and the reactance of the stator, R_2 and X_2 to those of the rotor and R_{eq} and X_{eq} to the equivalent parameters for the motor as a whole. V_1 is the voltage applied to the stator. From Thevenin theorem,

$$R_{eq} + iX_{eq} = (R_1 + iX_1) || R_C || iX_m \quad (1.1)$$

where $R_1 || R_2$ denotes the equivalent resistances of a parallel combination of R_1 and R_2, and

$$\tilde{V}_{eq} = \tilde{V}_1 \frac{R_C || iX_m}{R_1 + iX_1 + (R_C || iX_m)} \quad (1.2)$$

Considering the Thevenin equivalent circuit, the torque Γ is given by

$$\frac{3\frac{R_2}{s}|\tilde{V}_{eq}|^2}{[(R_{eq} + \frac{R_2}{s})^2 + (X_{eq} + X_2)^2]N_1} \quad (1.3)$$

Where N_1 is the frequency at which voltage is applied to the stator, N_2 is the spinning speed of the rotor and $s = \frac{N_2 - N_1}{N_1}$. Thus we see that the torque emerges in terms of the resistances and reactances of the various components. In this work, we will derive an expression for Γ based entirely on Maxwell's equations. Our derivation requires no further knowledge of electromagnetism than can be acquired from the standard undergraduate text by Griffiths[2]. In Section 2 we provide the formula and in Section 3 we show the relevance of our formula.

2. DERIVATION

We show the top and front views of a squirrel cage motor in Figure 2.

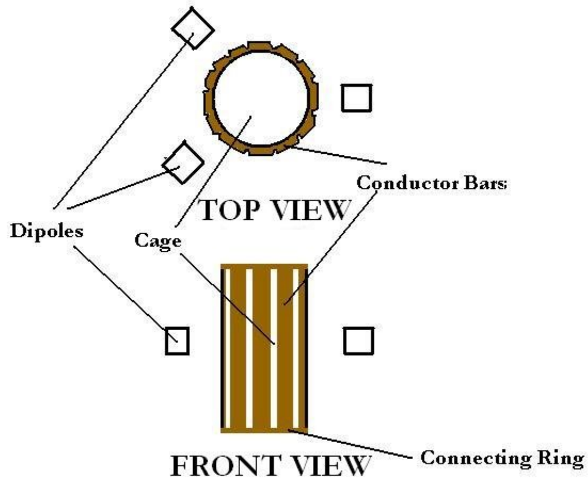

Figure 2 : Squirrel Cage Motor

The three dipoles, each assumed ideal, are mounted at equal angular spacing at a radius of R. The cage has a radius r and height h, the conductor bars have thickness τ and conductivity σ. The three dipoles are now fed the three phases of an alternating voltage, $V=V_0 \sin \Omega t$.

We now consider a point on a circle of radius r, at an angle θ from a reference line, which, without loss of generality we may select as the radius to Dipole 1. Further we say, again without loss of generality, that the current through Dipole 1 is $I_0 \sin \Omega t$, through Dipole 2 is $I_0 \sin (\Omega t + 2\pi/3)$ and through Dipole 3 is $I_0 \sin (\Omega t + 4\pi/3)$. We show a schematic diagram in Figure 3.

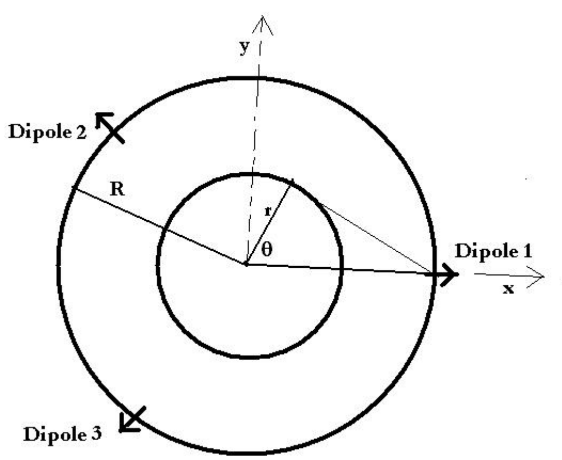

Figure 3 : The Point in Question

The formula for the field of a magnetic dipole is cumbersome so we go for the vector potential. The vector potential of a magnetic dipole of moment $\vec{m}$ is given by

$$\vec{A} = \frac{\mu_0}{4\pi}\frac{\vec{m}\times\vec{r}}{|\vec{r}|^3} \tag{2.1}$$

where r is the position vector of the field point relative to the dipole. At all points in the x-y plane A is in the z direction. For simplicity we use a two-dimensional geometry.

Now for a dipole of strength M at $\theta=0$,

$$\begin{aligned}\vec{r} &= -R\hat{i} + (r\cos\theta\hat{i} + r\sin\theta\hat{j})\\ \vec{m} &= M\hat{i}\end{aligned} \tag{2.2}$$

Writing $\vec{A}' = \dfrac{\vec{A}}{\mu_0/4\pi}$ (2.3)

we get

$$\vec{A}' = \frac{Mr\sin\theta}{(R^2 + r^2 - 2Rr\cos\theta)^{3/2}}\hat{k} \tag{2.4}$$

Likewise for the dipole at $\theta=2\pi/3$,

$$\vec{A}' = \frac{-\frac{1}{2}Mr\sin\theta - \frac{\sqrt{3}}{2}Mr\cos\theta}{[R^2 + r^2 - 2Rr\cos(\theta - 2\pi/3)]^{3/2}}\hat{k} \tag{2.5}$$

and for the dipole at $\theta=4\pi/3$

$$\vec{A}' = \frac{-\frac{1}{2}Mr\sin\theta + \frac{\sqrt{3}}{2}Mr\cos\theta}{[R^2 + r^2 - 2Rr\cos(\theta - 4\pi/3)]^{3/2}}\hat{k} \tag{2.6}$$

Expanding a term of the form $\dfrac{1}{(R^2 + r^2 - 2Rr\cos\alpha)^{3/2}}$ as $\dfrac{1}{R^3} + \dfrac{3r}{R^4}\cos\alpha + \ldots$

and substituting this in Eqs. (2.4) to (2.6) and adding them together we get

$$\vec{A}' = \frac{1}{R^3}\left(M_0 r\sin\theta - \frac{1}{2}M_1 r\sin\theta - \frac{\sqrt{3}}{2}M_1 r\cos\theta - \frac{1}{2}M_2 r\sin\theta + \frac{\sqrt{3}}{2}M_2 r\cos\theta\right)$$

$$+\frac{3r}{R^4}\left(M_0 r\sin\theta\cos\theta + ...\right) \tag{2.7}$$

where M_0, M_1 and M_2 are the strengths of Dipoles 1, 2 and 3 respectively.

Equation (2.7) has the form

$$\vec{A}' = \frac{1}{R^2}\left[\left(\frac{r}{R}\right)\left(M_0\sin\theta - \frac{1}{2}M_1\sin\theta - \frac{\sqrt{3}}{2}M_1\cos\theta - \frac{1}{2}M_2\sin\theta + \frac{\sqrt{3}}{2}M_2\cos\theta\right) + \left(\frac{r}{R}\right)^2\left(3M_0\sin\theta\cos\theta + ...\right)\right] \tag{2.8}$$

which is clearly a power series in r/R. We assume $r \ll R$ and retain only the first order terms.

Now using the fact that the dipole moment of an electromagnetic dipole is directly proportional to the current flowing through it, and letting M be the peak strength of each dipole in our motor,

$$M_0 = M\sin\Omega t$$

$$M_1 = M\sin(\Omega t + 2\pi/3$$

$$M_2 = M\sin(\Omega t + 4\pi/3) \tag{2.9}$$

Substituting (2.9) in the first term of (2.8) we get

$$\vec{A}' = \left\{\frac{rM}{R^3}\left[\sin\Omega t\sin\theta - \frac{1}{2}\left(-\frac{1}{2}\sin\Omega t + \frac{\sqrt{3}}{2}\cos\Omega t\right)\left(\sin\theta + \sqrt{3}\cos\theta\right) - \frac{1}{2}\left(-\frac{1}{2}\sin\Omega t - \frac{\sqrt{3}}{2}\cos\Omega t\right)\left(\sin\theta - \sqrt{3}\cos\theta\right)\right]\right\}\hat{k} \tag{2.10}$$

which on simplification reduces to

$$\vec{A} = -\frac{3\mu_0}{8\pi}\frac{rM}{R^3}\cos(\theta + \Omega t)\hat{k} \tag{2.11}$$

Using $B = \mathrm{curl}\ A$ and shifting to cylindrical co-ordinates $\boldsymbol{\rho}$, $\boldsymbol{\theta}$, z we get

$$\vec{B} = \frac{3\mu_0}{8\pi}\frac{M}{R^3}\sin(\theta+\Omega t)\hat{\rho} - \frac{3\mu_0}{8\pi}\frac{M}{R^3}\cos(\theta+\Omega t)\hat{\theta} \qquad (2.12)$$

Thus B is rotary and has an angular frequency Ω. From the geometry it is evident that the $\boldsymbol{\theta}$ component of B will not be responsible for inducing any emf in the cage, the $\boldsymbol{\rho}$ component alone will contribute. Now let ω be the rotation rate of the cage and set

$$B_0 = \frac{3\mu_0}{8\pi}\frac{M}{R^3} \qquad (2.13)$$

Qualitatively the rotating B drags the cage along with it so ω is clockwise. We now quantify this drag using techniques similar to those found in references (3) to (6). Considering a time t_0 and a point at an angle θ we go to a frame moving at speed Ωr in the -$\boldsymbol{\theta}$ direction. In this frame $B=B_0\sin(\theta+\Omega t_0)$ and is constant in time. Also the cage appears to move with speed $(\Omega-\omega)r$ in the $\boldsymbol{\theta}$ direction. Transferring to the frame of the cage, we get an induced electric field

$$\vec{E} = \vec{v}\times\vec{B} \quad = -B_0(\Omega-\omega)r\sin(\theta+\Omega t_0)\hat{z} \qquad (2.14)$$

Now we use $J=\sigma E$, remembering that this constitutive relation would be valid for low enough E which in turn is valid only for low values of Ω-ω, from Eq. (2.14).We also note that J can be assumed constant through an element of cross sectional area $\tau r d\theta$ (τ is the conductor thickness and $\tau \ll r$). We get the current as

$$dI = \sigma[-B_0(\Omega-\omega)r\sin(\theta+\Omega t_0)]\tau r d\theta \qquad (2.15)$$

and the infinitesimal force from $I\,l \times B$ is

$$d\vec{F} = \sigma h[-B_0(\Omega-\omega)r\sin(\theta+\Omega t_0)][B_0\sin(\theta+\Omega t_0)]\tau r d\theta\hat{\theta} \qquad (2.16)$$

from which the expression for the infinitesimal torque becomes

$$d\vec{N} = -\sigma h r^3 \tau B_0^2 (\Omega - \omega) \sin^2(\theta + \Omega t_0) d\theta \hat{z} \tag{2.17}$$

As B is rotary, the profiles of the field at two different times are identical except for a shift through a certain angle which becomes irrelevant when we integrate over the whole cage. Hence the torque does not vary with time so long as Ω and ω are constant and we may evaluate it by setting $t_0=0$ in (2.17) and integrating the same over the whole range of θ i.e $0\leq\theta\leq 2\pi$. Hence

$$\vec{N} = -\pi\sigma h r^3 \tau B_0^2 (\Omega - \omega) \hat{z} \tag{2.18}$$

which reduces to

$$\vec{N} = -\frac{9\mu_0^2 M^2 r^3 h\sigma\tau}{64\pi R^6} (\Omega - \omega) \hat{z} \tag{2.19}$$

If the cage has an iron core, then μ_0 in (18) will be replaced by μ_{eff} which will be somewhere between μ_0 and μ_{core}. The entire μ_{core} will not act as the dipoles themselves are mounted in air.

As expected the torque is in the direction of motion and is directly proportional to the defect of the rotational frequency over the excitation frequency. It should be noted that in the low frequency range, the Thevenin circuit answer in Eq. (1.3) also yields a torque proportional to Ω-ω. The torque as a function of the slip $s=1 - \omega/\Omega$ has been measured and one such measurement[7] is shown in Figure 4.

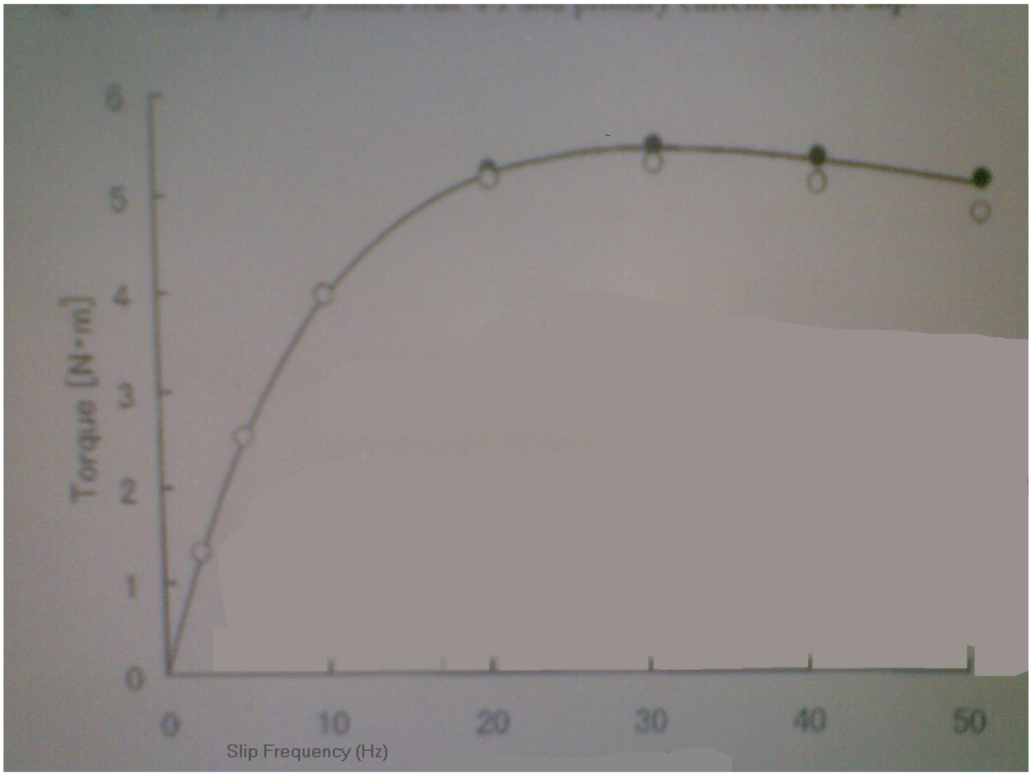

Figure 4 : The Experimental Measurement

As expected, the torque rises linearly in the low slip region. The setting in of the saturation at a slip frequency around 10 Hz is a consequence of the breakdown of the linear response (i.e. $J=\sigma E$). As E increases, the first correction to J appears in the form $J=\sigma E-\sigma'|E|^2E$. For practical applications, the motor generally operates in the linear regime.

3. Application

We apply this formula to a motor of type 6FXA7059 manufactured by Crompton Greaves based on a design by ABB and obtain an estimate for the torque from the data and the picture given on the company website, shown in Figure 5.

Figure 5 : Crompton Greaves Traction Motor of Type 6FXA7059

The data is as follows (continuous ratings) :

Number of dipoles : 6
Maximum voltage : 2180 V
Maximum current : 370 A
Speed : 1583 rpm
Weight : 2050 kg

We see from a second website[8] that the continuous rated torque is 6930 N m and also that the torque increases to 10000 N m when the current is jacked up to 450 A, which are the maximum ratings.

We start by noting from the picture that
h:R=2:1
R:r=3:1.
The current value indicates copper wire of gauge zero zero. Assuming the weight of the stator and rotor to be equal, we have approximately 240 metres of wire in each dipole. Further assuming a density of 6000 $kg\ m^{-3}$ for the laminated iron cage we get r=0.16 m, R=0.5 m and h=1 m. A third assumption takes the dipoles as squares of side 20 cm so that the 6 dipoles may cover a third of the circumference of the cage. Then each dipole gets about 360 turns of wire, and the dipolar strength is about 1450 $A\ m^2$. Taking μ_{eff} as 100 and considering 4 per cent slip operation at 25 Hz, the torque obtained is about 8000 N m. Thus we see this derivation has produced a good ballpark estimate for the torque, starting from information which would not have yielded any answer from the Thevenin formulation.

We end by noting an important feature of our result. The dipole moment $\mathbb{M}$ in Eq. (2.2) is directly proportional to the input current hence the torque as given by Eq. (2.19) depends quadratically on the input current. This is a result which is not contained in the Thevenin circuit analysis. We have tried to look for data with which we could test this finding. In reference (8) we find the data sets $\mathrm{I}=370\ \mathrm{A}$, $\mathrm{N}=6930\ \mathrm{N\ m}$ and $\mathrm{I}=450\ \mathrm{A}$, $\mathrm{N}=10000\ \mathrm{N\ m}$. While I changes by a factor of 1.22, the torque changes by a factor of 1.44 which is almost exactly what our Eq. (2.19) predicts.

4. ACKNOWLEDGEMENT

I am grateful to KVPY, Government of India, for a fellowship.

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